Self-organized criticality driven by deterministic rules

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We have investigated the essential ingredients allowing a system to show self-organized criticality (SOC) in its collective behavior. Using the Bak-Sneppen model of biological evolution as our paradigm, we show that the random microscopic rules of update can be effectively substituted with a chaotic map without changing the universality class. Using periodic maps SOC is preserved, but in a different universality class, as long as the spectrum of frequencies is broad enough. [S1063-651X(97)06410-6]

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Complex extended systems showing critical behavior, a lack of scale in their features, appear to be widespread in nature, being as diverse as earthquakes [1], creep phenomena [2], material fracturing [3–5], fluid displacement in porous media [6,7], interface growth [8,9], river networks [10–12], and biological evolution [13–16]. At variance with equilibrium statistical mechanics, these systems do not need any fine tuning of a parameter to be in a critical state. To explain this behavior, Bak, Tang, and Wiesenfeld introduced the concept of self-organized criticality (SOC) through the simple sandpile model [17,18].

In recent years, several models with extremal dynamics have been shown to exhibit SOC when noise is present [19]. In this Brief Report we show that for this class of systems, noise can be replaced by either a chaotic or a *quasiperiodic* signal without destroying criticality. To illustrate this point we consider, as an example, the model proposed by Bak and Sneppen to describe the coevolution of natural species [19]. The result that different, even deterministic, microscopic rules can induce SOC in the collective behavior of a population points to a greater relevance of SOC in nature.

In the Bak-Sneppen (BS) model an ecosystem is described by a one-dimensional lattice, every site of which is occupied by a species. Species with stronger mutual interactions in the ecosystem are arranged on nearest-neighbor sites (the lattice can be interpreted as a food chain or as a food web in more than one dimension). Each species is characterized by its fitness, describing the average number of offsprings an individual of that species can have in the given environment. This definition of the fitness also accounts for the greater resistance to mutations of fitter species since mutations must propagate over a greater number of individuals to become a genetic trait of the species. Thus the species with the lowest fitness is the one that feels the strongest evolutionary pressure [14–16]. Its fate is to either evolve or get extinct, and its place will be taken by some newcomer species in the same ecological niche. Therefore, the fitness of the species occupying that site is the most likely to change in a short time. The nearest-neighbor species will find a different environment and their fitnesses will result changed too. As a result of such a simple dynamical rule, the system exhibits sequences of causally connected evolutionary events called *avalanches* [19]. The number of avalanches N follows a power-law distribution

$$N(s)\sim s^{-\tau}, \tag{1}$$

where s is the size of the avalanche and $\tau\sim 1.07$ [20,21] is the avalanche critical exponent. This kind of behavior, which is the essence of self-organized criticality, has actually been observed in paleontological data [13], suggesting that evolution and extinction may be episodic at all scales (a feature that goes under the name of *punctuated equilibrium*) [14–16].

In nature, the evolution of the least-fit species is due to genetic mutation. In the BS model, this mutation is realized by giving to the corresponding species a random fitness. As shown in [19], each lattice site j is assigned a fitness, namely, a random number between 0 and 1. At each time step in the simulation the smallest fitness is found. Then the fitnesses of the minimum and of the two nearest neighbors are updated according to the rule

$$f_{n+1}=F(f_n) \tag{2}$$

that assigns a new fitness f_{n+1} at time $n+1$ to the chosen lattice site. Indeed, in the original BS model, the function F is just a random function with a uniform distribution between 0 and 1. The system reaches a stationary critical state in which the distribution of fitnesses is zero below a certain threshold $f_c\sim 0.667\,02$ [20,21] and uniform above it. It is also possible to define other quantities that show a power-law behavior with their own critical exponent. Prominent among them are the first and all return time distributions of activity (a site is defined as active when its fitness is the minimum one)

$$P_f(t)\sim t^{-\tau_f},\quad P_a(t)\sim t^{-\tau_a}, \tag{3}$$

where $\tau_f\sim 1.58$ and $\tau_a\sim 0.42$ [20–22].

Different microscopic rules have been proposed to describe how the mutation of the least-fit species induces mutations of the nearest neighbors [24]: Changing the microscopical dynamical rule affects the universality class of the model, but not its SOC property. This fact shows the robust-

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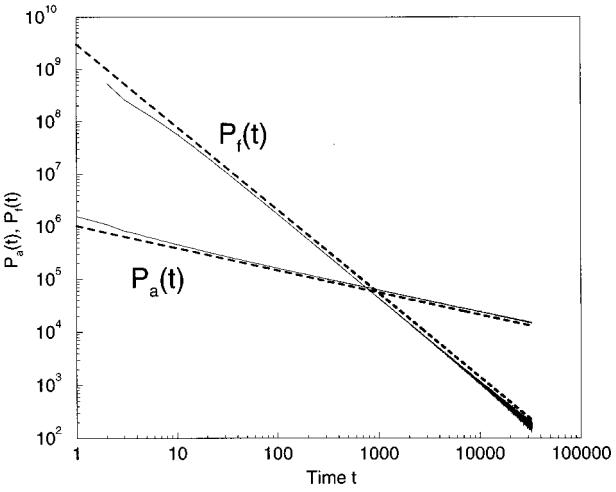

FIG. 1. First and all return distributions for a BS model with Bernoulli updating rule with $r=2$. The exponents are the same as for the BS model, namely, $\tau_f=1.58$ and $t_a=0.42$. For all the simulations shown here, we used a lattice of 2^{14} sites and 5×10^9 iterations exploiting the tree-algorithm explained in Ref. [26].

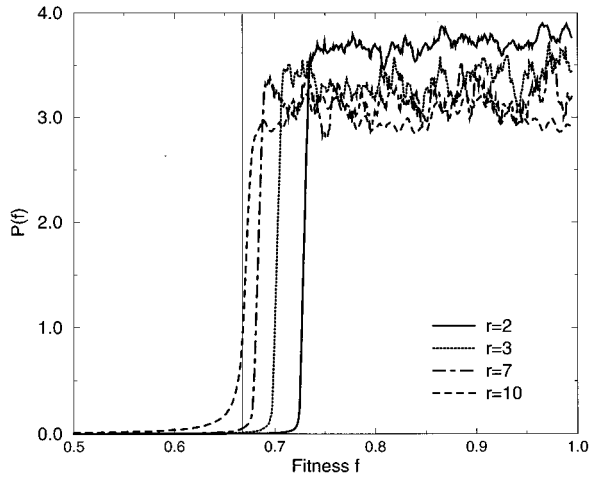

FIG. 2. Distribution of the fitnesses for $r=2,3,7,10$; the threshold for $r=2$ is quite different from the usual BS threshold, while the threshold corresponding to $r=10$ is very close to the BS value (given by the vertical line). For all the simulations shown here, we used a lattice of 2^{14} sites and 5×10^9 iterations.

ness of the SOC behavior and poses the question of what the minimal requirements the BS model has to satisfy are in order to be critical.

While in the BS model the updating rule (2) is a random function, one could consider, instead, deterministic updating. Indeed, we started by considering a deterministic rule, whose statistical properties resemble those of a stochastic function, namely, the Bernoulli map [25]

$$f_{n+1}=G_r(f_n)=[rf_n], \tag{4}$$

where $[f]$ stands for the value of f modulus 1 and $r\in\mathbb{N}$ is a constant. It has been shown (see [25] and references therein) that this map has a uniform invariant measure (for any integer value of r) and that the Lyapunov exponent λ is given by $\lambda=\ln r$. In Fig. 1 we show the power-law behavior of the first and all return probability distributions in the case $r=2$. The critical exponents τ, τ_a, and τ_f obtained coincide with those found in [19] for the random updating. Moreover, our simulations show that for all values of r the systems fall in the BS universality class, i.e., they all have the same critical exponents (see also [22]). The stationary distribution of the fitnesses, on the other hand, follows a different pattern. Indeed, Fig. 2 shows that the threshold for $r=2$ is bigger than the one found for the random case. On increasing the value of r, the threshold moves towards the BS value (see Fig. 2). For noninteger values of r ($r>1$), SOC is still preserved within the BS universality class. However, in this case, the distribution of the generated numbers is not uniform and consequently it influences the distribution of the fitnesses at the stationary state.

The next step is then to consider updating rules that can be tuned to chaotic behavior by changing a parameter. To that effect, we take as updating rule for the fitnesses the logistic or Feigenbaum map, namely,

$$f_{n+1}=\lambda f_n(1-f_n). \tag{5}$$

The reasons for studying this rule are manifold. On the one hand, this map has already been considered in the context of biological evolution models and population dynamics [27–29] and can thus provide a possible deterministic interpretation of the evolution inside every ecological niche. Moreover, it has been shown that it describes the behavior of a wide variety of systems in nature [30]. On the other hand, it has a regime in which it is chaotic as well as one in which it is not, depending on whether λ is bigger or less than the critical value $\lambda_\infty\sim3.569\,94$ [25] (for $\lambda>\lambda_\infty$ there are windows in which the map is periodic; in this paper we will take λ outside these windows). As Fig. 3 shows, for those values of λ for which the map is chaotic, the system not only exhibits SOC but also stays in the same universality class as the original BS model [22]. For $\lambda<\lambda_\infty$ we find that the system is not critical any more. Indeed, below λ_∞ every site follows a

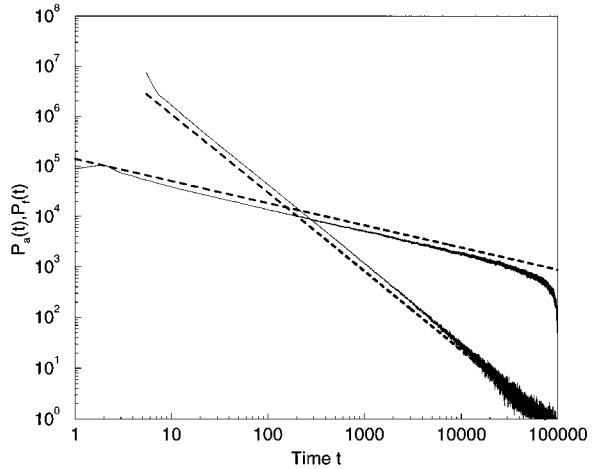

FIG. 3. First and all return distributions for a BS model with the logistic map with $\lambda=4$ as updating rule. The exponents are the same as for the BS model $\tau_f=1.58$ and $\tau_a=0.42$. In all the simulations shown in this figure, we used a lattice of 2^{13} sites and 10^8 iterations.

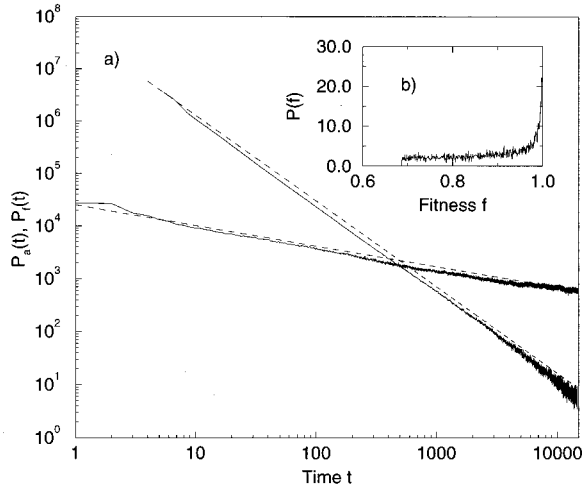

FIG. 4. (a) First and all return distributions for a BS model with disordered periodic updating rules; the exponents are $\tau_f = 1.65(1)$ and $\tau_a = 0.38(1)$. (b) Distribution of the fitnesses. In all the simulations shown in this picture, we used a lattice of 2^{12} sites and 5×10^9 iterations. The frequencies ω_i are uniformly distributed in the interval $[\pi, 40\pi]$.

periodic orbit and to understand this loss of criticality one needs to investigate the case of periodic updating rules in every site.

Let us then consider a model in which the choice of the new fitness is done according to the map

$$f_{n+1} = \frac{\sin[\arcsin(2f_n - 1) + \omega_j] + 1}{2}$$

$$= \frac{\sin[\omega_j(n+1) + \phi_j] + 1}{2}, \tag{6}$$

where the ω_j's and ϕ_j's are, in principle, different for each site j and n is the time step. In our calculations we have chosen the frequencies ω_j such that

$$\omega_j \neq \omega_i \tag{7}$$

and the phases ϕ_j to verify

$$\phi_j \neq \phi_i . \tag{8}$$

These two conditions are sufficient but not necessary. Indeed, the results shown here do not change if, by chance, Eqs. (7) and (8) are not fulfilled for some pair (i,j), as long as the distribution of frequencies is broad enough.

Bearing in mind that these maps are periodic, let us turn to Fig. 4, where the first and all return probabilities are shown. The universality class changes with respect to the original BS model, with $\tau_f = 1.65(1)$ and $\tau_a = 0.38(1)$, but the SOC behavior is preserved. These exponents will in general depend on the particular choice of frequency distribution (7). The range of frequencies chosen for Fig. 4 is just an example out of a wide range of possibilities for the frequency distribution. A more detailed analysis of the change in the universality class when varying the frequency range is a matter of current investigation and will be presented elsewhere. If instead of choosing all the frequencies different we choose $\omega_j = \omega$ and all phases different $\phi_j \neq \phi_i$ the SOC is destroyed (even if the fitnesses are organized above a threshold). This result sheds light on the loss of criticality for model (5) when $\lambda < \lambda_\infty$ because there the system reduces to a model in which the sites are updated according to a periodic rule where only condition (8) and not condition (7) is fulfilled.

It is worth noticing that even though chaotic maps produce series of numbers that may (statistically) resemble random numbers (with the exception of the functional form of the invariant density), the behavior of the systems feels the details of the underlying dynamics, as shown by the dependence of the threshold on the parameter r of the Bernoulli map. In particular, the Bernoulli map for $r = 2$ is formally equivalent to a coin toss [25,31] (the paradigm of randomness) and still the threshold is different. However, at variance with the case of pure random noise, in the Bernoulli map the correlations decay exponentially instead of being a δ function. Then the system feels the details of the rule and while staying in the same universality class exhibits a different threshold. One can then conclude that, provided the statistical properties are those of a random process, SOC will persist. These results are complementary to the ones obtained in Refs. [32,33]: They showed that a chaotic system can be used instead of a heat bath to obtain thermalization (the chaotic system is referred to as a ''booster'' [32,33]). This is analogous to what happens in the BS model: Noise (thermal or otherwise) can be replaced by a deterministic chaotic system without significant changes in the stationary state. However, stochasticity in the updating rule is sufficient but not necessary: SOC persists even in the absence of chaos for periodic updating rules.

The results shown here indicate that the feature that ensures SOC in systems with extremal dynamics is not the randomness of the actual updated value but the fact that the choice of the site where the change is going to be performed (namely, the minimum rule) is random. Moreover, as long as there is enough diversity among the species on the lattice, the longer the memory (or the internal correlation) of each member, the higher the threshold. Indeed, in the case of chaotic maps, the diversity is ensured by the random assignation of the initial values and as much as the chaoticity is increased we see that the threshold decreases. The extremal case is being given by the BS model.

In the case of the periodic map instead, the random initial conditions are not enough to ensure diversity. Thus, in order to have SOC, we have to choose at random also the internal time scale, i.e., the periods. Summarizing, SOC can emerge in a fully deterministic model in the presence of disorder.

These results add strength to the relevance of SOC in physics and biology since they allow different microscopic mechanisms to underlie its appearance as a collective behavior. From the point of view of biological evolution, this result could also account for less wild variations of the fitnesses.

At this stage, several questions arise. What is the mechanism that allows the system to recognize the underlying dynamics? Moreover, when the microscopic rule is disordered periodic, does the distribution of the frequencies influence the universality class of the model? These questions are, we think, of utmost importance in order to understand the widespread appearance of self-organized criticality in nature.

Note added. After finishing this work we became aware of Ref. [34], in which a similar line of reasoning is pursued for the effects of disorder on a population of integrate-and-fire oscillators.

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[1] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).
[2] S. I. Zaitsev, Physica A **189**, 411 (1992).
[3] A. Petri, G. Paparo, A. Vespignani, A. Alippi, and M. Costantini, Phys. Rev. Lett. **73**, 3423 (1994).
[4] P. Diodati, F. Marchesoni, and S. Piazza, Phys. Rev. Lett. **67**, 2239 (1991).
[5] G. Caldarelli, F. di Tolla, and A. Petri, Phys. Rev. Lett. **77**, 2503 (1996).
[6] D. Wilkinson and J. F. Willemsen, J. Phys. A **16**, 3365 (1983).
[7] M. Cieplak and M. O. Robbins, Phys. Rev. Lett. **60**, 2042 (1988).
[8] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).
[9] K. Sneppen, Phys. Rev. Lett. **71**, 101 (1993). ,
[10] A. Rinaldo, I. Rodriguez-Iturbe, R. Rigon, E. Ijjasz-Vasquez, and R. L. Bras, Phys. Rev. Lett. **70**, 822 (1993).
[11] A. Maritan, F. Colaiori, A. Flammini, M. Cieplak, and J. R. Banavar, Science **272**, 984 (1996).
[12] G. Caldarelli, A. Giacometti, A. Maritan, I. Rodriguez-Iturbe, and A. Rinaldo, Phys. Rev. E **55**, R4865 (1997).
[13] M. D. Raup, Science **251**, 1530 (1986).
[14] N. Eldredge and S. J. Gould, *Punctuated Equilibria: An Alternative to Phyletic Gradualism in Models in Paleobiology* (Freeman, Cooper and Co., San Francisco, 1972).
[15] S. J. Gould and N. Eldredge, Paleobiology **3**, 115 (1977).
[16] N. Eldredge and S. J. Gould, Nature (London) **332**, 211 (1988).
[17] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[18] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[19] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[20] S. Maslov, M. Paczuski, and P. Bak, Phys. Rev. Lett. **73**, 2162 (1994).
[21] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[22] The two exponents τ_f and τ_a shown in Eq. (3) are not independent and have to sum to 2 in the one-dimensional model. On the other hand, also the exponents τ in Eq. (1) and, say, τ_a are not independent since they are related through Maslov's equation [23]. In this light, τ_a can be considered as the only independent exponent of the model and its computation sufficient to check the universality class.
[23] S. Maslov, Phys. Rev. Lett. **77**, 1182 (1996).
[24] M. Vendruscolo, P. De Los Rios, and L. Bonesi, Phys. Rev. E **54**, 6053 (1996).
[25] H. G. Schuster, *Deterministic Chaos*, 2nd ed. (VCH-Verlag, Weinheim, 1989).
[26] P. Grassberger, Phys. Lett. A **200**, 277 (1995).
[27] R. M. May, Science **186**, 645 (1974).
[28] R. M. May, Nature (London) **261**, 459 (1976).
[29] R. M. May and G. F. Oster, Am. Nat. **110**, 573 (1976).
[30] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Boston, 1980).
[31] J. Ford, Phys. Today **36** (4), 40 (1993).
[32] M. Bianucci, L. Bonci, G. Trefan, B. West, and P. Grigolini, Phys. Lett. A **174**, 377 (1993).
[33] M. Bianucci, B. West, and P. Grigolini, Phys. Lett. A **190**, 447 (1994).
[34] A. Corral, C. J. Pérez, and A. Díaz-Guilera, Phys. Rev. Lett. **78**, 1492 (1997).